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New similarity solutions for fragmenting systems with continuous loss of mass

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Abstract. In a recent article, Saied and El-Wakil discussed the similarity solutions of a fragmenting system. Their article is an extension of our own work on an equation including continuous loss of mass. Saied and El-Wakil restrict their considerations to similarity solutions on a special choice of parameters. We demonstrate here that the symmetry group given in Saied and El-Wakil's article is incomplete. Furthermore, we are able to construct new similarity solutions derived from the missing subgroup.

In their article, Saied and El-Wakil [1] considered a special case of the discrete fragmentation model originally introduced by McGrady and Ziff [2] which was extended by a continuous mass loss. The similarity solutions of a continuous fragmentation model were first discussed by Baumann *et al* [3]. In the work of Saied and El-Wakil, the symmetry structure of the extended fragmentation model was analysed using Lie's method.

According to [1], this fragmentation model can be described by the equation

$$\partial_{ut} w(u,t) + A\partial_{uu} w(u,t) + B\partial_u w(u,t) + Cw(u,t) = 0 \tag{1}$$

where

$$A = -\eta u^{\beta+1}$$

$$B = -\eta(2\beta + 2 + \mu)u^{\beta} + u^{\beta}$$

$$C = (\beta + 2 - \eta\beta(\beta + 1 + \mu))u^{\beta-1}$$

are constants which are related to the fragmentation rate, the distribution of the daughter-particle mass and the continuous mass loss.

This equation was derived from an integrodifferential equation of the continuous loss of mass which has its origin in [2, 4, 5]. This type of differential equation (1) was also introduced by Edwards *et al* [6] discussing fragmentation with mass loss. Differentiating this integrodifferential equation with respect to the variable u, we get differential equation (1). u denotes the spatial and t the time variable in equation (1).

The similarity solutions discussed in [1] are restricted to a special case of the physical parameter $\beta = \delta + 1$. The solutions follow from a subgroup which linearly combines two vector fields of a three-dimensional symmetry group. With this reduction, the final solution is represented either by Kummer's equation or by confluent hypergeometric equations. In the case without continuous mass loss, a scaling solution is described in analogy to [3].

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In our examination of (1), we used the MATHEMATICA program Lie, which was written to analyse the symmetries of ordinary differential equations (ODE) and partial differential equations (PDE) [7, 8]. The program Lie uses the standard procedure of Lie's method, as described by Olver [9], to derive the determining equations of the infinitesimal transformations. The latest version of the program includes a routine which automatically calculates the exact analytical solution of the determining equations [3]. This routine contains a repeated integration and back-substitution procedure to solve various systems of PDE. First, the determining equations are decoupled and expressed in a standard form, equivalent to the previous form, by an algorithm described in [8, 10]. The standard form of the determining equations is solved by various techniques such as separation of variables and introduction of potential functions for further simplification. If an equation cannot be explicitly integrated, the equation and the corresponding functions are added to the final result. With the help of this program, we discovered that the symmetry group described in [1] is incomplete. The resulting infinitesimals, which are the complete solution of the determining equations under the restriction $\beta = \delta + 1$, are given by

$$\xi^u = c_4(u^{1-\beta} - 2\beta\eta t u) - c_3\frac{u}{\beta} \tag{2}$$

$$\xi^t = c_1 + c_3 t + c_4\beta^2\eta t^2 \tag{3}$$

$$\phi^w = c_2 w + c_4\Gamma t w + f(u,t) \tag{4}$$

with

$$\Gamma = \beta(-1 + \eta + \beta\eta + \eta\mu). \tag{5}$$

The arbitrary function $f(u,t)$ has to solve the original equation (1). This property reflects the linearity of the PDE and should be expected. Apart from the infinite-dimensional subgroup reflecting the linearity of equation (1), the symmetry group contains a finite four-dimensional subgroup because of the four integration constants.

Comparing our results with those given in [1], we have that the symmetry group in that article is a subgroup of symmetries (2)–(4) with $f(u,t) = c_4 = 0$. Thus, the similarity solutions described in [1] are also incomplete. We will demonstrate that our symmetry reduction for subgroup c_4 will give a new type of solution in the time- and space-dependent case. In the case of subgroup c_4, our results differ from those presented in [1], where only the scaling solution in the space variable is considered.

We found an additional completely new solution which is characterized by the group constant c_4 and, therefore, we have a discrete four-dimensional Lie-algebra instead of a three-dimensional one. According to the labelling used in [1], a basis of this discrete Lie-algebra is chosen:

$$X_1 = \partial_t \tag{6}$$

$$X_2 = w\partial_w \tag{7}$$

$$X_3 = -\frac{u}{\beta}\partial_u + t\partial_t \tag{8}$$

$$X_4 = (u^{1-\beta} - 2\beta\eta t u)\partial_u + \beta^2\eta t^2\partial_t + \Gamma t w\partial_w. \tag{9}$$

Consequently, the tables for the commutators and the adjoint representation have to be generalized. The commutator table is given in table 1 and the adjoint representation in table 2. The adjoint representation is calculated according to the procedure described by Olver [9]. First, following the method proposed by Olver, the adjoint representation of the Lie-algebra (6)–(9) given in table 2 is calculated. With the aid of the adjoint-representation transformations generated by the vector fields, (6)–(9) can be found to reduce a vector field of the general form

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$$

to the following six cases which can be summarized in the optimal system:

$$X_4 \qquad X_3 + aX_2 \text{ with } a \in R \qquad X_2 + X_1 \qquad X_2 - X_1 \qquad X_2 \qquad X_1.$$

To compute invariant solutions of the symmetry group of equation (1), only these six cases need be considered. The last five cases correspond to solutions which have already been calculated by Saied and El-Wakil [1]. If we wish to calculate an invariant solution containing X_4 in a linear combination with the other basis elements according to the optimal system, it is sufficient to calculate the invariant solution corresponding to X_4.

Table 1. Commutators for the Lie-algebra of the rate equation (1) where Γ is given by equation (5).

[,]	X_1	X_2	X_3	X_4
X_1	0	0	X_1	$2\beta^2\eta X_3 + \Gamma X_2$
X_2	0	0	0	0
X_3	$-X_1$	0	0	X_4
X_4	$-\Gamma X_2 - 2\beta^2\eta X_3$	0	$-X_4$	0

Table 2. Adjoint representation of the Lie-algebra where Γ is given by equation (5).

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$X_3 - \epsilon X_1$	$X_4 - \epsilon(\Gamma X_2 + 2\beta^2\eta X_3) + \epsilon^2\beta^2\eta X_1$
X_2	X_1	X_2	X_3	X_4
X_3	$e^{\epsilon}X_1$	X_2	X_3	$e^{-\epsilon}X_4$
X_4	$X_1 + \epsilon(\Gamma X_2 + 2\beta^2\eta X_3) + \epsilon^2\beta^2\eta X_4$	X_2	$X_3 + \epsilon X_4$	X_4

A solution of (1) which is invariant under the action of X_4 has to satisfy the invariant surface condition, a PDE of first order for $w(u, t)$:

$$(u^{1-\beta} - 2\beta\eta t u)w_u + \beta^2\eta t^2 w_t - \Gamma t w = 0. \tag{10}$$

Solving the linear PDE (10), we get the similarity representation of $w(u, t)$:

$$w(u, t) = W(z)t^{\nu} \tag{11}$$

where the similarity variable combines the time and space variables in a non-trivial way

$$z = \frac{1}{\beta\eta t^2 u^{\beta} - t}.$$

The scaling exponent of the time γ is given as

$$\gamma = \frac{-1 + \eta + \beta\eta + \eta\mu}{\beta\eta}.$$

Inserting the similarity solution (11) into the original PDE, we obtain an ODE for $W(z)$:

$$bW(z) + czW'(z) + dz^2W''(z) = 0 \tag{12}$$

where

$$b = 2 + \beta - \beta\eta - \beta^2\eta - \beta\eta\mu$$
$$c = -\beta + \beta\eta + \beta^2\eta + \beta\eta\mu$$
$$d = -\beta^2\eta$$

are constants. It is obvious that the solutions of equation (12) are powers in z

$$W(z) = k_1 z^{\kappa_1} + k_2 z^{\kappa_2}. \tag{13}$$

The scaling exponents κ_1 and κ_2 of the similarity variable z have to solve a quadratic equation in the parameters β, η and μ:

$$\kappa_{1,2} = \frac{-1 + \eta + 2\beta\eta + \eta\mu \pm \sqrt{(1 - \eta - 2\beta\eta + \eta\mu)^2 - 4\eta(-2 - \beta + \beta\eta + \beta^2\eta + \beta\eta\mu)}}{2\beta\eta}.$$

The reduced equation (12) has the same form as the ODE (35) in [1]. Compared with our discussion, Saied and El-Wakil restrict themselves to the time-independent case with the similarity variable $z = u$. Thus, their similarity solution $w(u, t) = F(u)$ is completely independent of t. In contrast to these results, our similarity variable $z = (\beta\eta t^2 u^\beta - t)^{-1}$ and our solution $w(u, t) = t^\gamma W(z)$ depend explicitly on the time t and, therefore, our solution describes a space- and time-dependent process.

For $t = 0$, our solution to equation (1) has a singularity. For general values of the parameters β, η and μ, solution (13) is only defined in regions where z is greater than zero. At the borders of these regions, there also exist singularities with respect to the determining equation

$$u^\beta = \frac{1}{\beta\eta t}.$$

In conclusion, we have demonstrated that the special case $\beta = \delta + 1$, discussed by Saied and El-Wakil [1], allows a four-dimensional discrete symmetry group as well as an infinite-dimensional continuous group. We have derived a completely new and exact solution of the fragmentation equation (1) which has not been calculated in [1]. Our solution $w(u, t)$ can be expressed in a simple form, i.e. in powers of the similarity variable $z = (\beta\eta t^2 u^\beta - t)^{-1}$.

Finally, we note that the order of the symmetry group is reduced from a four-dimensional to a two-dimensional group if we consider the case in which the parameters β, δ, μ and η are independent. In this case, the resulting group is a subgroup of (2)–(4) where $c_4 = c_3 = 0$. This means that the equation is invariant with respect to a time translation and a scaling of the dependent variable and that it shows the expected linear behaviour.

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